THE UNIVERSAL HOPF CYCLIC THEORY

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1. Introduction

In noncommutative geometry, as the category of algebras of various flavors replaced the category of spaces of various flavors, Hopf algebras arose as the natural candidate to study the symmetries of a noncommutative space. Unlike the classical notion of symmetry, the notion of noncommutative symmetry has four different types:

`[MC]`	**module coalgebra**	A Hopf algebra acting on a coalgebra in a compatible way.
`[CA]`	**comodule algebra**	A Hopf algebra coacting on a algebra in a compatible way.
`[MA]`	**module algebra**	A Hopf algebra acting on a algebra in a compatible way.
`[CC]`	**comodule coalgebra**	A Hopf algebra coacting on a coalgebra in a compatible way.

These compatibility conditions can be expressed concisely as the (co)multiplication structure morphism of the corresponding (co)algebra being a B-(co)module morphism where B is our base Hopf algebra. We are interested in such symmetries in the context of cyclic (co)homology. In the sequel the term "cyclic theory" will mean a functor from a suitable category of algebras into the category of (co)cyclic k-modules and the term "cyclic (co)homology" will mean a suitable (co)homology functor from the category of (co)cyclic k-modules into the category of k-modules.

Since Connes and Moscovici's seminal work on the cyclic cohomology of Hopf algebras and transverse index theorem [3, 4], there has been a surge in the interest in such symmetries [5, 7, 14, 15, 16, 17]. However, a (co)cyclic theory and basic tools of cyclic cohomology had to be built from scratch for each type of symmetry separately [6, 9, 10]. To complicate the story further, the cyclic dual of each (co)cyclic theory constructed thus far is non-trivial in stark contrast with the ordinary non-equivariant case where the dual is always trivial. This means there are 8 potentially different types of cyclic theories in the presence of a Hopf symmetry. Considering the variety of (co)homology functors one can apply to a given (co)cyclic object, we suddenly see a phletora of cyclic cohomology theories which are sensitive to the Hopf symmetry in the noncommutative universe.

Amazingly, Khalkhali and Rangipour [11] showed that if we view a Hopf algebra as a coalgebra enjoying an `[MC]`-type symmetry over itself, then the cyclic dual of the canonical cocyclic object is functorially isomorphic to the canonical cyclic object of the same Hopf algebra viewed as an algebra enjoying an `[CA]`-type symmetry over itself. This result suggests that there is a deep meta-symmetry lurking behind, connecting all of these different types of cyclic theories.

In a previous paper [10], Khalkhali and the author successfully unified the `[MA]` and `[MC]`-type cyclic theories and their cyclic duals under the banner of bivariant Hopf-cyclic cohomology. In this paper, we aim to unravel further the meta-symmetry behind all of these cyclic theories and construct a new universal cyclic theory encompassing all types of symmetries we stated above, agreeing with, and extending further, the previous definitions given in the literature. The construction is purely categorical and each individual theory is obtained by modifying certain parameters. These parameters, or variables, are (i) a symmetric monoidal category which will stand for the category modules over a ground ring k, (ii) a class of morphisms called *transpositions* which will play the role of a coefficient, (iii) an arbitrary exact comonad which will replace a k-flat Hopf algebra and finally (iv) a suitable category of (co)algebras called *transpositive (co)algebras* which will play the role of (co)module (co)algebras.

One *practical* consequence of this formal exercise in category theory is that we no longer need to define a different theory for each type of symmetry and then prove directly that it really is cyclic, which is quite technical and involved. The recipe we provide in this paper ensures that the end object is not only equivariantly (co)cyclic but also the right object for all known cases. The results of this article will give us the licence to ignore the technical problems of existence of a right kind of cyclic theory and to engage with more pressing questions such as excision, Morita invariance and homotopy invariance in the presence of a Hopf symmetry. Moreover, now that the these cyclic theories are defined by universal properties we expect such questions to become more accessible for investigation.

Here is a plan of this paper. In Section 2 we give definitions of transpositions and transpositive (co)algebras in an arbitrary symmetric strict monoidal category $\mathcal{C}$. In the same section we also describe ordinary B-(co)module (co)algebras over an arbitrary bialgebra B as transpositive algebras in a specific monoidal category with respect to certain classes of transpositions. In Section 3 we construct the universal para-(co)cyclic theory for the category of transpositive (co)algebras. Next in Section 4, we incorporate an arbitrary exact comonad $\mathtt{B}$ into our machinery. In this section, specifically in Theorem 4.8, we show that every pseudo-para-(co)cyclic $\mathtt{B}$-comodule admits an *approximation* (Definition 4.1) in the category of (co)cyclic $\mathtt{B}$-comodules. For an arbitrary bialgebra B, in Section 5 we recover the Hopf cyclic and equivariant cyclic theories of B-module (co)algebras [10] and bialgebra cyclic theory of B-comodule algebras [9]. The key observation we use is that the universal para-cyclic theory actually takes values in the category of pseudo-para-(co)cyclic B-modules in these cases. As a side result, we recover the Hopf–Hochschild homology [8] by using the techniques developed in this paper. We end the paper by defining the missing cyclic theory for comodule coalgebras as a natural extension of the cyclic theories defined hitherto.

Throughout this article, we assume $\mathcal{C}$ is a small category. If we require $\mathcal{C}$ to be monoidal $\otimes$ will denote the monoidal product of $\mathcal{C}$ and we will assume $(\mathcal{C}, \otimes)$ is a symmetric strict monoidal category with a unit object I.

1.1. **Acknowledgements.** We trace the genesis of this article back to a series of discussions the author had with Krzysztof Worytkiewicz on the formal properties of the Hopf cyclic (co)homology.

2. Transpositions and transpositive (co)algebras

In this section we will use a rudimentary version of "calculus of braid diagrams" in monoidal categories as developed in [13].

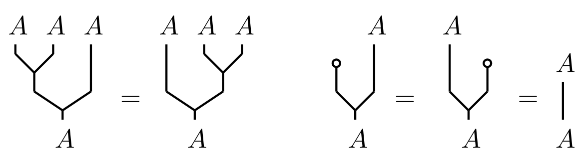
Definition 2.1. Fix an object M in $\mathcal{C}$ and a unique morphism $w_{M,X} : M \otimes X \to X \otimes M$ for each object X chosen from a subset $\mathcal{T}$ of $Ob(\mathcal{C})$. The datum $(M, \mathcal{T}, \{w_{M,X}\}_{X \in \mathcal{T}})$ is called *a class of transpositions* and is denoted by w. For such an object $X \in \mathcal{T}$, the morphism $w_{M,X}$ and its inverse $w^{-1}_{M,X}$ if it exists, are going to be denoted by

respectively. We do not require transpositions to be invertible, nor do we require the inverses to be transpositions themselves even if they exist.

An object A in $\mathcal{C}$ is called an algebra if there exist morphisms $A^{\otimes 2} \xrightarrow{\mu_A} A$ and $I \xrightarrow{e} A$ such that the following diagrams commute

$$\begin{tikzcd} A^{\otimes 3} \arrow[r, "\mu_A \otimes A"] \arrow[d, "A \otimes \mu_A"'] & A^{\otimes 2} \arrow[d, "\mu_A"] \\ A^{\otimes 2} \arrow[r, "\mu_A"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "I \otimes e"] \arrow[d, "e \otimes I"'] \arrow[dr, "id"] & A^{\otimes 2} \arrow[d, "\mu_A"] \\ A^{\otimes 2} \arrow[r, "\mu_A"'] & A \end{tikzcd}$$

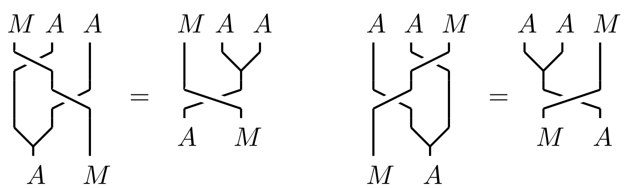
In other words, A is a monoid object in $(\mathcal{C}, \otimes)$, or equivalently both $(A \otimes \ \cdot\ )$ and $(\ \cdot\ \otimes A)$ are a monads in $\mathcal{C}$. A coalgebra $(C, \delta_C, \varepsilon)$ in $\mathcal{C}$ is simply an algebra in $\mathcal{C}^{op}$. These conditions for an algebra A will also be denoted by the following diagrams



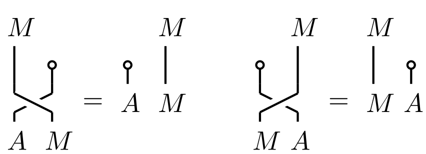
Definition 2.2. An algebra (A, μ_A, e) in $\mathcal{C}$ is called a *w-transpositive algebra* if there exists a morphism $w_{M,X} \colon M \otimes X \to X \otimes M$ in w such that the following diagram commutes

$$\begin{tikzcd} M \otimes A \otimes A \arrow[r, "w_{M,A} \otimes A"] \arrow[d, "M \otimes \mu_A"'] & A \otimes M \otimes A \arrow[r, "A \otimes w_{M,A}"] & A \otimes A \otimes M \arrow[d, "\mu_A \otimes M"] \\ M \otimes A \arrow[rr, "w_{M,A}"] & & A \otimes M \\ & M \arrow[ul, "M \otimes e"] \arrow[ur, "e \otimes M"'] & \end{tikzcd}$$

One can compare this diagram with the "bow-tie" diagram of entwining structures [1, Diagram 5.5]. However, there one requires M to be a coalgebra and one has similar compatibility conditions on the comultiplication structure. Here we do not require M to be a coalgebra. The interaction between the multiplication morphism and $w_{M,A}$, and its inverse $w_{M,A}^{-1}$ if it exits, will be denoted by



Similarly, the interaction between the unit morphism and $w_{M,A}$, and $w_{M,A}^{-1}$ if it exits, will be denoted by



A w-transpositive coalgebra $(C, \delta_C, \varepsilon)$ is a w^{op}-transpositive algebra in the opposite monoidal category $(\mathcal{C}^{op}, \otimes^{op})$ where $U \otimes^{op} V := V \otimes U$ for any two objects $U, V \in Ob(\mathcal{C}^{op})$.

For the examples we are going to consider below, we fix a commutative associative unital ring k. Our base symmetric monoidal category is $\mathbf{Mod}(k)$ the category of k-modules with $\otimes_k$ the ordinary tensor product over k taken as the monoidal product $\otimes$. We also assume B is an associative/coassociative unital/counital bialgebra, or a Hopf algebra with an invertible antipode whenever it is necessary.

Example 2.3. Fix a right/left B-module/comodule (M, β_M, λ_M). We let $w_{M,X} : M \otimes X \to X \otimes M$ be a transposition if (i) (X, α_X) is a left B-module and (ii) $w_{M,X}$ is defined by the formula

$$w_{M,X}(m \otimes x) := m_{(-1)}x \otimes m_{(0)} = (\alpha_X \otimes M) \circ (B \otimes s_{M,X}) \circ (\lambda_M \otimes X)(m \otimes x)$$

for any $m \otimes x \in M \otimes X$ where $s_{M,X}$ is the ordinary switch morphism. For this class of transpositions w, an algebra (A, μ_A, e) is w-transpositive if A is a left B-module algebra. Similarly $(C, \delta_C, \varepsilon)$ is a w-transpositive coalgebra, if C is a right B-comodule coalgebra.

Example 2.4. Fix a left/right B-module/comodule (M, α_M, ρ_M). We let $w_{M,X} : M \otimes X \to X \otimes M$ be a transposition if (i) (X, β_X) is a right B-comodule and (ii) $w_{M,X}$ is defined by the formula

$$w_{M,X}(m \otimes x) := x_{(0)} \otimes m x_{(1)} = (\alpha_X \otimes M) \circ (s_{M,X} \otimes B) \circ (M \otimes \beta_X)(m \otimes x)$$

for any $m \otimes x \in M \otimes X$ where $s_{M,X}$ is the ordinary switch morphism. For this class of transpositions w, an algebra (A, μ_A, e) is w-transpositive if A is a right B-comodule algebra. Similarly $(C, \delta_C, \varepsilon)$ is a w-transpositive coalgebra, if C is a left B-module coalgebra.

3. The universal para-(co)cyclic theory

Definition 3.1. Let S be the category with objects $\{0,1\}$ where there is one unique morphism $i \to j$ between any two objects $i, j \in \{0,1\}$. A functor $F\colon S \to \mathcal{C}$ will be called an S-module.

Lemma 3.2. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two groupoids which have the property that between any two objects there is a unique morphism. Let $F, G\colon \mathcal{G} \longrightarrow \mathcal{C}$ and $F'\colon \mathcal{G}' \longrightarrow \mathcal{C}$ be three arbitrary functors and let $g, h \in \mathrm{Ob}(\mathcal{G})$ and $g' \in \mathrm{Ob}(\mathcal{G}')$ be three arbitrary objects.*

(1) *Any morphism $F(g) \xrightarrow{u} G(h)$ in $\mathcal{C}$ can be lifted to a natural transformation of functors of the form $F \xrightarrow{u} G$.*
(2) *Any morphism $F(g) \xrightarrow{v} F'(g')$ in $\mathcal{C}$ can be lifted to a morphism in $\mathcal{C}$ of the form $\mathrm{colim}_{\mathcal{G}} F \xrightarrow{v} \mathrm{colim}_{\mathcal{G}'} F'$*

Proof. We will denote the unique morphism between from an object x to another object y in $\mathcal{G}$ by $x \xrightarrow{\alpha_{y,x}} y$. Let $F(g) \xrightarrow{u} G(h)$ be an arbitrary morphism in $\mathcal{C}$ and define

$$u_x := G(\alpha_{x,h}) \circ u \circ F(\alpha_{g,x})$$

for any $x \in Ob(\mathcal{G})$. In order $u.$ to define a natural transformation, for any $x, y \in Ob(\mathcal{G})$ one must have

$$u_y \circ F(\alpha_{y,x}) = G(\alpha_{y,x}) \circ u_x$$

So we check

$$\begin{aligned} u_y \circ F(\alpha_{y,x}) =& G(\alpha_{y,h}) \circ u \circ F(\alpha_{g,y}) \circ F(\alpha_{y,x}) \\ =& G(\alpha_{y,x}) \circ G(\alpha_{x,h}) \circ u \circ F(\alpha_{g,x}) = G(\alpha_{y,x}) \circ u_x \end{aligned}$$

for any $x, y \in Ob(\mathcal{G})$ as we wanted to show. This finishes the first part of the assertion.

For the second part, let $F(x) \xrightarrow{\phi(x)} \mathrm{colim}_{\mathcal{G}} F$ and $F'(x') \xrightarrow{\phi'(x')} \mathrm{colim}_{\mathcal{G}'} F'$ be the structure morphisms of the corresponding colimits. Then one has morphisms

$$\psi(x) := \phi'(g') \circ v \circ F(\alpha_{g,x})$$

of the form $x \xrightarrow{\psi(x)} \mathrm{colim}_{\mathcal{G}'} F'$ for any $x \in Ob(\mathcal{G})$. We check that

$$\psi(x) \circ F(\alpha_{x,y}) = \phi'(g') \circ v \circ F(\alpha_{g,x}) \circ F(\alpha_{x,y}) = \phi'(g') \circ v \circ F(\alpha_{g,y}) = \psi(y)$$

for any $x, y \in Ob(\mathcal{G})$ meaning there is a unique morphism $\mathrm{colim}_{\mathcal{G}} F \to \mathrm{colim}_{\mathcal{G}'} G$ which, by abuse of notation, we still denote by v. □

Definition 3.3. Let C and M be two arbitrary objects in $\mathcal{C}$ such that we have a transposition $M \otimes C \xrightarrow{w_{M,C}} C \otimes M$. For every $n \geqslant 0$, we define an S-module $P_n(C, M)$ in $\mathcal{C}$ as follows: let $P_n(C, M)$ is the functor from S to $\mathcal{C}$ given on the objects by

$$P_n(C,M)(0) := C^{\otimes n} \otimes M \otimes C \qquad P_n(C,M)(1) := C^{\otimes n+1} \otimes M \tag{3.1}$$

Moreover,

$$P_n(C,M)(0\to 1) := \Big(C^{\otimes n}\otimes M\otimes C \xrightarrow{t_{n+2}} C^{\otimes n+1}\otimes M\Big) \tag{3.2}$$

is the cyclic permutation coming from the symmetric monoidal structure of $\mathcal{C}$ thus its inverse provides $P_n(C,M)(1\to 0)$.

Definition 3.4. Let Λ be Connes' cyclic category [2] and $\Lambda_{\mathbb{N}}$ and $\Lambda_{\mathbb{Z}}$ be the variations of Λ as defined in [10]. Let us recall the presentation we will use in this paper: the category $\Lambda_{\mathbb{N}}$ has objects $[n]$ indexed by natural numbers $n \geqslant 0$ and is generated by morphisms $[n] \xrightarrow{\partial_j^n} [n+1]$, $[n+1] \xrightarrow{\sigma_i^n} [n]$ and $[n] \xrightarrow{\tau_n^\ell} [n]$ with $0\leqslant j\leqslant n+1$, $0\leqslant i\leqslant n$ and $\ell\in\mathbb{N}$. These generators are subject to the following relations

$$\begin{aligned}
\partial_i^{n+1}\partial_j^n =& \partial_{j+1}^{n+1}\partial_i^n \quad\text{and}\quad \sigma_j^{n-1}\sigma_i^n = \sigma_i^{n-1}\sigma_{j+1}^n \text{ for } i\leqslant j \quad\text{and } \tau_n^s\tau_n^t = \tau_n^{s+t} \text{ for } s,t\in\mathbb{N}\\
\sigma_i^n\partial_i^n =& \sigma_i^n\partial_{i+1}^n = \tau_n^0 \quad\text{and}\quad \partial_i^n\sigma_j^n = \begin{cases}\sigma_{j+1}^{n+1}\partial_i^{n+1} & \text{if } i\leqslant j\\ \sigma_j^{n+1}\partial_{i+1}^{n+1} & \text{if } i>j\end{cases}\\
\partial_j^n\tau_n^i =& \tau_{n+1}^{i+p}\partial_q^n \text{ where } (i+j) = (n+1)p+q \text{ with } 0\leqslant q\leqslant n\\
\tau_n^i\sigma_j^n =& \sigma_q^n\tau_{n+1}^{i+p} \text{ where } (-i+j) = (n+1)(-p)+q \text{ with } 0\leqslant q\leqslant n
\end{aligned}$$

The category $\Lambda_{\mathbb{Z}}$ is an extension of $\Lambda_{\mathbb{N}}$ where we allow morphisms of the form τ_n^i with $i\in\mathbb{Z}$. Then Λ is a quotient of $\Lambda_{\mathbb{Z}}$ where we put the extra relations $\tau_n^{n+1} = id_n$ for $n\geqslant 0$. The category Λ_+ is the subcategory of $\Lambda_{\mathbb{N}}$ generated by ∂_j^n and σ_i^n with only $0\leqslant i\leqslant n$ and $0\leqslant j\leqslant n$. A functor $F\colon \Lambda \longrightarrow \mathcal{C}$ will be referred as *a cocyclic module in* $\mathcal{C}$ while any functor of the form $F\colon \Lambda_{\mathbb{N}}\longrightarrow\mathcal{C}$ or $F\colon\Lambda_{\mathbb{Z}}\longrightarrow\mathcal{C}$ will be referred as *a para-cocyclic module in* $\mathcal{C}$. A (para-)cyclic module F in $\mathcal{C}$ is defined to be a (para-)cocyclic module in $\mathcal{C}^{op}$. A morphism between (para-)cocyclic modules $h\colon F\longrightarrow G$ in $\mathcal{C}$ is just a natural transformation of functors.

Theorem 3.5. *Let (C,δ_C,ε) be a w-transpositive coalgebra. Let $\operatorname{colim}_S P_\bullet(C,M)$ be level-wise colimit of $P_\bullet(C,M)$. Then $\operatorname{colim}_S P_\bullet(C,M)$ carries a para-cocyclic module structure.*

Proof. The cosimplicial structure morphisms are given by

$$\partial_i := C^{\otimes i}\otimes\delta_C\otimes C^{\otimes n-i}\otimes M \quad\text{and}\quad \sigma_j := C^{\otimes j+1}\otimes\varepsilon\otimes C^{\otimes n-1-j}\otimes M$$

which are defined only for $0\leqslant i\leqslant n$ and $0\leqslant j\leqslant n-1$ and on $C^{\otimes n+1}\otimes M$. We also let

$$\partial_{n+1} := (C^{\otimes n}\otimes w_{M,C}\otimes C)\circ(C^{\otimes n}\otimes M\otimes\delta_C)$$

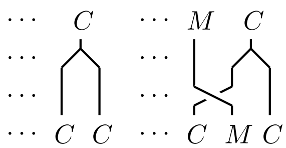
which is a morphism defined on $C^{\otimes n}\otimes M\otimes C$. The fact that the morphisms ∂_i and σ_j for $0\leqslant i\leqslant n+1$ and $0\leqslant j\leqslant n$ are well-defined on the level-wise colimits follows from Lemma 3.2. The para-cocyclic structure morphisms are already used in this definition since we are going to define

$$\tau_n := C^{\otimes n}\otimes w_{M,C}$$

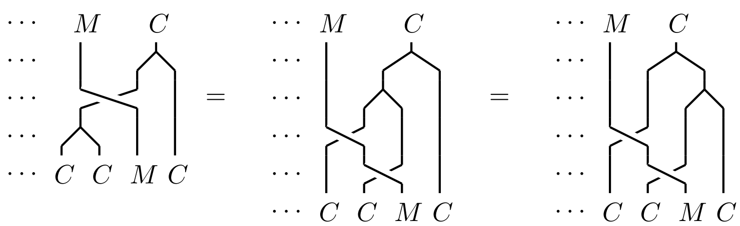
for any $n \geqslant 0$. This is a morphism of the form

$$P_n(C, M)(0) \xrightarrow{\tau_n} P_n(C, M)(1)$$

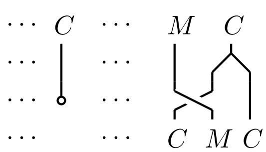
The fact that τ_n is well-defined on $\operatorname{colim}_S P_n(A, M)$ for any $n \geqslant 0$ again is a consequence of Lemma 3.2. The verification of the cosimplicial identities between ∂_i and σ_j for the range $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$ is standard and follows from the fact that C is a coassociative counital coalgebra in $\mathcal{C}$. Next, we consider $\partial_j \partial_{n+1}$. If $0 \leqslant j \leqslant n$, one can describe the composition by



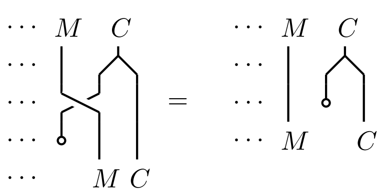
This shows $\partial_j \partial_{n+1} = \partial_{n+2} \partial_j$ for $0 \leqslant j \leqslant n$. For $j = n+1$, by using the fact that C is a w-transpositive coalgebra we see that $\partial_{n+1} \partial_{n+1}$ can be described as



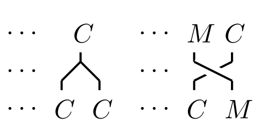
which is equivalent to saying $\partial_{n+1} \partial_{n+1} = \partial_{n+2} \partial_{n+1}$. This finishes the proof that $\operatorname{colim}_S P_\bullet(C, M)$ is pre-cosimplicial. Now we consider $\sigma_i \partial_{n+1}$. If $0 \leqslant i < n$, the composition can be described by



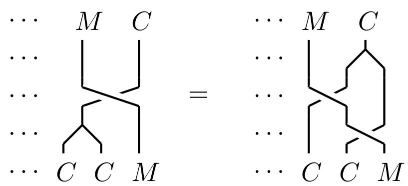
Then one can easily see that $\sigma_i \partial_{n+1} = \partial_n \sigma_i$ for $0 \leqslant i < n$. We also observe that $\sigma_n \partial_{n+1} = id$ since



This finishes the proof that $\operatorname{colim}_S P_\bullet(C, M)$ is a cosimplicial object in $\mathcal{C}$. Now we must check the para-cocyclic identities. First we observe that $\tau_{n+1} \partial_0 = \partial_{n+1}$ by definition. Next, we consider $\tau_{n+1} \partial_i$. For the range $0 < i < n$ we represent the composition by



which means one has $\tau_{n+1}\partial_i = \partial_{i-1}\tau_n$ for the range $0 < i < n$. For $i = n$ we consider $\partial_n\tau_n$ which is represented by



which is equivalent to saying $\partial_n\tau_n = \tau_{n+1}^2\partial_0 = \tau_{n+1}\partial_{n+1}$. So far we have the following relations

$$\partial_i\tau_n = \tau_{n+1}\partial_{i+1} \text{ for } 0 \leqslant i < n \text{ and } \partial_n\tau_n = \tau_{n+1}^2\partial_0$$

Using these relations one can show that

$$\partial_i\tau_n^j = \tau_{n+1}^{j+p}\partial_q \text{ where } (i+j) = (n+1)p + q$$

for any $n \geqslant 0$, $0 \leqslant i \leqslant n+1$ and $j \geqslant 0$ i.e. $\mathrm{colim}_S P_\bullet(C,M)$ is a pre-para-cocyclic object in $\mathcal{C}$. We leave the verification of the identities

$$\tau_n^j\sigma_i = \sigma_q\tau_{n+1}^{i+p} \text{ where } (i-j) = (n+1)(-p) + q$$

involving para-cyclic operators and the codegeneracy operators to the reader. $\square$

For simplicity, the para-cocyclic module $\mathrm{colim}_S P_\bullet(C,M)$ will be denoted by $T_\bullet(C,M)$.

4. Approximation theorems for pseudo-para-(co)cyclic objects

Definition 4.1. Here we assume $\mathcal{C}$ is an arbitrary small category and let $\mathcal{D}$ be a subcategory. For an arbitrary object X of $\mathcal{C}$ a morphism $\mathrm{App}_{\mathcal{D}}(X) \xrightarrow{u_X} X$ is called the approximation of X within $\mathcal{D}$ if (i) $\mathrm{App}_{\mathcal{D}}(X)$ is an object in $\mathcal{D}$ and (ii) every morphism $D \xrightarrow{v} X$ with $D \in Ob(\mathcal{D})$ factors *uniquely* through u_X, i.e. there exists a *unique* morphism $D \xrightarrow{v'} \mathrm{App}_{\mathcal{D}}(X)$ such that $v = u_X \circ v'$. Similarly, the co-approximation $\mathrm{CoApp}_{\mathcal{D}}(X)$ is the approximation of X within $\mathcal{D}^{op}$ viewed as an object of $\mathcal{C}^{op}$. We do not make any assumptions on the existence of (co)approximations.

Theorem 4.2. *Assume $\mathcal{C}$ is a small category with equalizers. Then the approximation $\mathrm{App}_\Lambda(X_\bullet)$ of any para-(co)cyclic module $X_\bullet$ within the category of (co)cyclic modules in $\mathcal{C}$ exists.*

Proof. Every para-(co)cyclic object has a canonical endomorphism $\omega_\bullet$ defined at each degree $n \geqslant 0$ by $\omega_n := \tau_n^{n+1}$ which commutes with all the structure morphisms. The cyclic approximation $\mathrm{App}_\Lambda(X_\bullet)$ of a para-(co)cyclic object $X_\bullet$ is defined degree-wise as the equalizer of the pair $(\omega_\bullet, id_\bullet)$ of para-(co)cyclic modules in $\mathcal{C}$. Since both $\omega_\bullet$ and $id_\bullet$ are morphisms of para-(co)cyclic module in $\mathcal{C}$, their equalizer $\mathrm{App}_\Lambda(X_\bullet) \to X_\bullet$ is a morphism of para-(co)cyclic modules in $\mathcal{C}$. Moreover, $\tau_n^{n+1} = id_n$ on $\mathrm{App}_\Lambda(X_n)$, i.e. $\mathrm{App}_\Lambda(X_\bullet)$ is a (co)cyclic module in $\mathcal{C}$. Assume we have a morphism $Y_\bullet \xrightarrow{f_\bullet} X_\bullet$ of para-(co)cyclic modules

in $\mathcal{C}$ where $Y_\bullet$ is a (co)cyclic module in $\mathcal{C}$. Since $\omega_n f_n = \tau_n^{n+1} f_n = f_n \tau_n^{n+1} = f_n$ for any $n \geqslant 0$, $f_\bullet$ factors through the equalizer $\mathrm{App}_\Lambda(X_\bullet)$. □

Definition 4.3. Recall from [12] that an endo-functor $\mathtt{B}\colon \mathcal{C} \to \mathcal{C}$ is called a comonad if there exist natural transformations $\mathtt{B} \xrightarrow{\Delta} \mathtt{B}^2$ and $\mathtt{B} \xrightarrow{\varepsilon} id_{\mathcal{C}}$ which fit into commutative diagrams

$$\begin{tikzcd} \mathtt{B}^2(X) \arrow[r, "\mathtt{B}(\Delta_X)"] & \mathtt{B}^3(X) \\ \mathtt{B}(X) \arrow[u, "\Delta_X"] \arrow[r, "\Delta_X"'] & \mathtt{B}^2(X) \arrow[u, "\Delta_{\mathtt{B}(X)}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathtt{B}^2(X) \arrow[r, "\mathtt{B}(\varepsilon_X)"] & \mathtt{B}(X) & \mathtt{B}^2(X) \arrow[l, "\varepsilon_{\mathtt{B}(X)}"'] \\ & \mathtt{B}(X) \arrow[ul, "\Delta_X"] \arrow[u, "id"] \arrow[ur, "\Delta_X"'] & \end{tikzcd}$$

However, we will diverge from the standard conventions and we will refer an object X as a $\mathtt{B}$-comodule if there exists a morphism $X \xrightarrow{\rho_X} \mathtt{B}(X)$ such that the following diagrams commute

$$\begin{tikzcd} \mathtt{B}(X) \arrow[r, "\Delta_X"] & \mathtt{B}^2(X) \\ X \arrow[u, "\rho_X"] \arrow[r, "\rho_X"'] & \mathtt{B}(X) \arrow[u, "\mathtt{B}(\rho_X)"'] \end{tikzcd} \qquad \begin{tikzcd} \mathtt{B}(X) \arrow[r, "\varepsilon_X"] & X \\ & X \arrow[ul, "\rho_X"] \arrow[u, "id_X"'] \end{tikzcd}$$

Such objects are called $\mathtt{B}$-coalgebras in [12] but later we will work with (co)algebras in the category of $\mathtt{B}$-comodules and it would have been awkward to call them "$\mathtt{B}$-coalgebra coalgebras". Also, a morphism $X \xrightarrow{f} Y$ between two $\mathtt{B}$-comodules is called a morphism of $\mathtt{B}$-comodules if one has a commuting diagram of the form

$$\begin{tikzcd} X \arrow[r, "\rho_X"] \arrow[d, "f"'] & \mathtt{B}(X) \arrow[r, "\varepsilon_X"] \arrow[d, "\mathtt{B}(f)"] & X \arrow[d, "f"] \\ Y \arrow[r, "\rho_Y"'] & \mathtt{B}(Y) \arrow[r, "\varepsilon_Y"'] & Y \end{tikzcd}$$

The full subcategory of $\mathtt{B}$-comodules in $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathtt{B}}$ and the category of $\mathtt{B}$-comodules and their morphisms is denoted by $\mathbf{CoMod}(\mathtt{B})$.

Example 4.4. Let $(\mathcal{C}, \otimes)$ be the category of k-modules with ordinary tensor product of modules as the monoidal product. Then any k-coalgebra (C, Δ, ε) defines two comonads $(\,\cdot\, \otimes C)$ and $(C \otimes\, \cdot\,)$. Moreover, the category of comodules in these cases are the same as the category of right and left C-comodules respectively.

Example 4.5. Let $(\mathcal{C}, \otimes)$ be the opposite category of k-modules with ordinary tensor product of modules as the monoidal product. Then any k-algebra (A, μ, e) determines two comonads $(\,\cdot\, \otimes A)$ and $(A \otimes\, \cdot\,)$. Moreover, the category of comodules with respect to these comonads are the same as the category of right and left A-modules respectively.

Definition 4.6. A comonad $\mathtt{B}$ is called left exact (resp. right exact) if $\mathtt{B}$ commutes with arbitrary small limits (resp. colimits). In other words for any functor $F\colon \mathcal{I} \to \mathcal{C}$ one has

$$\lim_{\mathcal{I}}(\mathtt{B} \circ F) \cong \mathtt{B}\,(\lim_{\mathcal{I}} F) \qquad \big(\text{resp. } \mathrm{colim}_{\mathcal{I}}(\mathtt{B} \circ F) \cong \mathtt{B}\,(\mathrm{colim}_{\mathcal{I}} F)\,\big)$$

And a comonad is called exact if it is both left and right exact.

Definition 4.7. Let $\mathtt{B}$ be a comonad on a category $\mathcal{C}$. A para-(co)cyclic object $T_\bullet\colon \Lambda_{\mathbb{N}} \to \mathcal{C}^{\mathtt{B}}$ is called a pseudo-para-(co)cyclic $\mathtt{B}$-comodule if its restriction to the subcategory Λ_+ factors through $\mathbf{CoMod}(\mathtt{B})$.

Theorem 4.8. *Let $\mathtt{B}$ be a left exact comonad on a complete category $\mathcal{C}$. Then every pseudo-para-cyclic $\mathtt{B}$-comodule $T_\bullet\colon \Lambda_{\mathbb{N}}^{op} \to \mathcal{C}^{\mathtt{B}}$ admits an approximation $\mathrm{App}_\Lambda(T_\bullet^{\mathtt{B}})$ within the category of cyclic $\mathtt{B}$-comodules.*

Proof. We are going to abuse the notation and use ∂_j, σ_i and τ_n^ℓ to denote $T(\partial_j^n)$, $T(\sigma_i^n)$ and $T(\tau_n^\ell)$ respectively. For every $n \geqslant 0$, denote the $\mathtt{B}$-comodule structure morphisms $T_n \to \mathtt{B}(T_n)$ by ρ_n. For any $n \geqslant 0$, define $T_n^m \xrightarrow{\eta_{n,m}} T_n$ as the equalizer of the pair of morphisms $\mathtt{B}(\tau_n^m)\rho_n$ and $\rho_n\tau_n^m$ for every $m \in \mathbb{N}$. Now define

$$T_n^{\mathtt{B}} := \lim_{m\in\mathbb{N}}\ T_n^m \xrightarrow{\eta_{n,m}} T_n$$

where $T_n^{\mathtt{B}} \xrightarrow{\eta_n} T_n$ is the canonical morphism into T_n for any $n \geqslant 0$. Consider the following non-commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
\mathtt{B}(T_n) \arrow[r, "\mathtt{B}(\tau_n^j)"] & \mathtt{B}(T_n) \arrow[r, "\mathtt{B}(\tau_n^i)"] & \mathtt{B}(T_n) \\
T_n \arrow[u, "\rho_n"] \arrow[r, "\tau_n^j"'] & T_n \arrow[u, "\rho_n"'] \arrow[r, "\tau_n^i"'] & T_n \arrow[u, "\rho_n"'] \\
T_n^{\mathtt{B}} \arrow[u, "\eta_n"] & T_n^{\mathtt{B}} \arrow[u, "\eta_n"'] &
\end{tikzcd}$$

Since η_n is the equalizer of the pairs of morphisms $\big(\rho_n\tau_n^i,\ \mathtt{B}(\tau_n^i)\rho_n\big)$ for all $i \in \mathbb{N}$, if we can show that

$$\rho_n\tau_n^i\tau_n^j\eta_n = \mathtt{B}(\tau_n^i)\rho_n\tau_n^j\eta_n \tag{4.1}$$

for all $i \in \mathbb{N}$ we will obtain a functorial 'restriction' of τ_n^j to $T_n^{\mathtt{B}}$ which will be denoted by $(\tau_n^j)^{\mathtt{B}}$ for any $j \in \mathbb{N}$. Consider the left hand side of Equation 4.1 which is

$$\rho_n\tau_n^{i+j}\eta_n = \mathtt{B}(\tau_n^{i+j})\rho_n\eta_n = \mathtt{B}(\tau_n^i)\mathtt{B}(\tau_n^j)\rho_n\eta_n = \mathtt{B}(\tau_n^i)\rho_n\tau_n^j\eta_n$$

as we wanted to show.

Now, for $0 \leqslant j \leqslant n+1$ consider the following diagram in $\mathcal{C}$

$$\begin{tikzcd}
\mathtt{B}(T_{n+1}) \arrow[r, "\mathtt{B}(\partial_j)"] & \mathtt{B}(T_n) \arrow[r, "\mathtt{B}(\tau_n^i)"] & \mathtt{B}(T_n) \\
T_{n+1} \arrow[u, "\rho_{n+1}"] \arrow[r, "\partial_j"'] & T_n \arrow[u, "\rho_n"'] \arrow[r, "\tau_n^i"'] & T_n \arrow[u, "\rho_n"'] \\
T^{\mathtt{B}}_{n+1} \arrow[u, "\eta_{n+1}"] & T^{\mathtt{B}}_n \arrow[u, "\eta_n"'] &
\end{tikzcd}$$

where the square on top right does not commute and square on top left commutes as long as $0 \leqslant j \leqslant n$. However, since $\partial_{n+1} = \partial_0 \tau_{n+1}$ (recall that $T_\bullet$ is cyclic not cocyclic) and τ_{n+1} has a restriction to $T^{\mathtt{B}}_{n+1}$, one can assume WLOG that $0 \leqslant j \leqslant n$. If we can show that

$$\rho_n \tau_n^i \partial_j \eta_{n+1} = \mathtt{B}(\tau_n^i)\rho_n \partial_j \eta_{n+1} \tag{4.2}$$

for any $i \in \mathbb{N}$, one obtains a unique morphism $T^{\mathtt{B}}_{n+1} \to T^{\mathtt{B}}_n$ which is going to be denoted as $(\partial_j)^{\mathtt{B}}$. The uniqueness of this morphism implies its functoriality. Consider the left hand side of the Equation 4.2

$$\rho_n \tau_n^i \partial_j \eta_{n+1} = \rho_n \partial_q \tau_{n+1}^{i+p} \eta_{n+1}$$

where $(i+j) = (n+1)p + q$ and $0 \leqslant q \leqslant n$. Now use the fact that $0 \leqslant j \leqslant n$ and $T_\bullet$ is a pseudo-para-cyclic to deduce

$$\begin{aligned}
\rho_n \partial_q \tau_{n+1}^{i+p} \eta_{n+1} =& \mathtt{B}(\partial_q)\rho_n \tau_{n+1}^{i+p} \eta_{n+1} = \mathtt{B}(\partial_q)\mathtt{B}(\tau_{n+1}^{i+p})\rho_n \eta_{n+1} \\
=& \mathtt{B}(\tau_{n+1}^{i})\mathtt{B}(\partial_j)\rho_n \eta_{n+1} = \mathtt{B}(\tau_{n+1}^{i})\rho_n \partial_j \eta_{n+1}
\end{aligned}$$

as we wanted to show. One can similarly prove that the relevant diagrams commute for the degeneracy morphisms. This finishes the proof that $T^{\mathtt{B}}_\bullet$ is a para-cyclic module in $\mathcal{C}$.

Now, for an arbitrary $j \in \mathbb{N}$ consider the non-commutative diagram

$$\begin{tikzcd}
T_n \arrow[r, "\rho_n"] & \mathtt{B}(T_n) \arrow[r, "\mathtt{B}(\rho_n)"] & \mathtt{B}^2(T_n) \\
T_n \arrow[u, "\tau_n^j"] \arrow[r, "\rho_n"'] & \mathtt{B}(T_n) \arrow[u, "\mathtt{B}(\tau_n^j)"'] \arrow[r, "\mathtt{B}(\rho_n)"'] & \mathtt{B}^2(T_n) \arrow[u, "\mathtt{B}^2(\tau_n^j)"'] \\
T^{\mathtt{B}}_n \arrow[u, "\eta_n"] & \mathtt{B}(T^{\mathtt{B}}_n) \arrow[u, "\mathtt{B}(\eta_n)"'] &
\end{tikzcd}$$

and the composition

$$\begin{aligned}
\mathtt{B}^2(\tau_n^j)\mathtt{B}(\rho_n)\rho_n\eta_n =& \mathtt{B}^2(\tau_n^j)\Delta_{T_n}\rho_n\eta_n = \Delta_{T_n}\mathtt{B}(\tau_n^j)\rho_n\eta_n \\
=& \Delta_{T_n}\rho_n\tau_n^j\eta_n = \mathtt{B}(\rho_n)\rho_n\tau_n^j\eta_n \\
=& \mathtt{B}(\rho_n)\mathtt{B}(\tau_n^j)\rho_n\eta_n
\end{aligned}$$

The equality of the first and the last terms implies $\rho_n\eta_n$ factors through the limit of the equalizers of the pairs $\mathtt{B}^2(\tau_n^j)\mathtt{B}(\rho_n)$ and $\mathtt{B}(\rho_n)\mathtt{B}(\tau_n^j)$ as j runs through the set of all natural numbers. But $\mathtt{B}$ is a left exact comonad which means this limit is exactly $\mathtt{B}(T_n^{\mathtt{B}})$. Thus we get the $\mathtt{B}$-comodule structure on $T_n^{\mathtt{B}}$ which implies $T_\bullet^{\mathtt{B}}$ is a para-cyclic module in $\mathcal{C}^{\mathtt{B}}$.

Now we need to show given any morphism $[n] \xrightarrow{\phi} [m]$ in $\Lambda_{\mathbb{N}}$ its image $\phi^{\mathtt{B}}$ under the newly constructed functor $T_\bullet^{\mathtt{B}}$ is a morphism of $\mathtt{B}$-comodules. In order to prove this fact we need the following diagram to commute

$$\begin{tikzcd}
\mathtt{B}(T_n^{\mathtt{B}}) & \mathtt{B}(T_m^{\mathtt{B}}) \arrow[l, "\mathtt{B}(\phi)"'] \\
T_n^{\mathtt{B}} \arrow[u, "\rho_n"] & T_m^{\mathtt{B}} \arrow[l, "\phi^{\mathtt{B}}"] \arrow[u, "\rho_m"']
\end{tikzcd}$$

To achieve this, first we need to show that the larger squares in the following diagrams commute

$$\begin{tikzcd}
\mathtt{B}(T_n) & \mathtt{B}(T_{n+1}) \arrow[l, "\mathtt{B}(\partial_j)"'] \\
T_n \arrow[u, "\rho_n"] & T_{n+1} \arrow[l, "\partial_j"'] \arrow[u, "\rho_n"'] \\
T_n^{\mathtt{B}} \arrow[u, "\eta_n"] & T_{n+1}^{\mathtt{B}} \arrow[l, "(\partial_j)^{\mathtt{B}}"] \arrow[u, "\eta_n"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathtt{B}(T_n) \arrow[r, "\mathtt{B}(\sigma_j)"] & \mathtt{B}(T_{n+1}) \\
T_n \arrow[u, "\rho_n"] \arrow[r, "\sigma_j"] & T_{n+1} \arrow[u, "\rho_m"'] \\
T_n^{\mathtt{B}} \arrow[u, "\eta_n"] \arrow[r, "(\sigma_j)^{\mathtt{B}}"'] & T_{n+1}^{\mathtt{B}} \arrow[u, "\eta_n"']
\end{tikzcd}$$

for any $i \geqslant 0$ and $0 \leqslant j \leqslant n$. In these diagrams, the top squares commute since $T_\bullet$ is pseudo-para-cyclic. We already have shown the bottom squares commute. Thus both diagrams commute for the prescribed range. Then we must show that the larger square in the following diagram commutes

$$\begin{tikzcd}
\mathtt{B}(T_n) \arrow[r, "\mathtt{B}(\tau_n^i)"] & \mathtt{B}(T_n) \\
T_n \arrow[u, "\rho_n"] \arrow[r, "\tau_n^i"'] & T_n \arrow[u, "\rho_n"'] \\
T_n^{\mathtt{B}} \arrow[u, "\eta_n"] \arrow[r, "(\tau_n^i)^{\mathtt{B}}"'] & T_n^{\mathtt{B}} \arrow[u, "\eta_n"']
\end{tikzcd}$$

The bottom square commutes while the top square does not. However, η_n equalizes $\rho_n\tau_n^i$ and $\mathtt{B}(\tau_n^i)\rho_n$. Therefore the larger diagram commutes. This finally finishes the proof that $T_\bullet^{\mathtt{B}}$ is a para-cyclic $\mathtt{B}$-comodule. Now we use Theorem 4.2 to finish the proof. □

Remark 4.9. There are 8 versions of Theorem 4.8

> A pseudo-para-(co)cyclic $\mathtt{B}$-(co)module in $\mathcal{C}$ admits an(a) (co)approximation in the category of (co)cyclic $\mathtt{B}$-(co)modules.

However, since the proof is given for an arbitrary complete category, by assuming $\mathcal{C}$ is both complete and cocomplete, one can use $\mathcal{C}$ and $\mathcal{C}^{op}$ interchangibly. This reduces the number of versions to 4:

> A pseudo-para-(co)cyclic $\mathtt{B}$-comodule in $\mathcal{C}$ admits an(a) (co)approximation in the category of (co)cyclic $\mathtt{B}$-comodules.

From the remaining 3, we are interested in the following

Theorem 4.10. *Let $\mathtt{B}$ be a left exact comonad on a complete category $\mathcal{C}$. Then every pseudo-para-cocyclic $\mathtt{B}$-comodule $T_\bullet\colon \Lambda_\mathbb{N} \to \mathcal{C}^\mathtt{B}$ admits an approximation $\mathrm{App}_\Lambda(T_\bullet^\mathtt{B})$ within the category of cocyclic $\mathtt{B}$-comodules.*

Proof. As before let ρ_n denote the $\mathtt{B}$-comodule structure morphism on T_n for any $n \geqslant 0$. Let $\Gamma(n)$ be the set of pairs of morphism of the form

$$\left(\rho_n\tau_n^i,\ \mathtt{B}(\tau_n^i)\rho_n\right) \text{ for } i \geqslant 0 \quad \text{or} \quad \left(\rho_n\partial_{n+1},\ \mathtt{B}(\partial_{n+1})\rho_n\right)$$

and we define $T_n^\gamma \xrightarrow{\eta(\gamma)} T_n$ as the equalizer of a pair $\gamma \in \Gamma(n)$. Next we define the approximation $T_n^\mathtt{B}$ for each $n \geqslant 0$ as

$$T_n^\mathtt{B} := \lim{}_\gamma\ T_n^\gamma \xrightarrow{\eta(\gamma)} T_n$$

where we use $T_n^\mathtt{B} \xrightarrow{\eta_n} T_n$ to denote the canonical morphism into T_n. The rest of the proof is very similar to that of Theorem 4.8 and we leave it to the interested reader to finish it. □

Definition 4.11. The (co)cyclic $\mathtt{B}$-comodule $\mathrm{App}_\Lambda(T_\bullet^\mathtt{B})$ corresponding to a pseudo-para-(co)cyclic $\mathtt{B}$-comodule $T_\bullet$ is called the universal (co)cyclic $\mathtt{B}$-comodule of $T_\bullet$. Moreover, given a functor of the form $\mathcal{F}\colon \mathcal{C}^\mathtt{B} \to \mathbf{Mod}(k)$ and a (co)homology functor $\mathcal{H}_*$ on the category of (co)cyclic k-modules, one can compute

$$\mathcal{H}_*\mathcal{F}(\mathrm{App}_\Lambda(T_\bullet^\mathtt{B}))$$

We will call this (co)homology as *the $\mathtt{B}$-equivariant $\mathcal{H}$-(co)homology of $T_\bullet$ with coefficients in $\mathcal{F}$.*

5. The universal cyclic theory of (co)module (co)algebras

5.1. **Hopf and equivariant cyclic theory of module coalgebras.** Fix a commutative unital ring k and an associative/ coassociative unital/counital k-bialgebra $(B, \mu_B, \mathbb{I}, \Delta_B, \varepsilon)$. Our base category is the opposite category of k-modules with the opposite tensor product over k, i.e. $(\mathcal{C}, \otimes) := (\mathbf{Mod}(k)^{op}, \otimes_k^{op})$. Our base comonad in $\mathcal{C}$ is going to be $\mathtt{B} := (\,\cdot \otimes B)$. Since we defined the comonad in the opposite category, we will use the algebra structure on B.

The category of left B-modules (i.e. $\mathtt{B}$-comodules in $\mathcal{C}$) is a monoidal category with respect to the ordinary tensor product of k-modules with the diagonal action of B on the left. Explicitly, given a pair of B-modules X and Y, the B-module structure on the product is given by

$$b(x \otimes y) := b_{(1)}x \otimes b_{(2)}y$$

for any $x \otimes y \in X \otimes Y$. However, the product is not symmetric unless B is cocommutative but there is a braided monoidal structure if one restricts oneself to use Yetter-Drinfeld modules. If denote the full subcategory of left B-modules of $\mathbf{Mod}(k)$ by $\mathcal{L}(B)$ then one can see that $\mathcal{C}^{\mathtt{B}} = \mathcal{L}(B)^{op}$.

Fix a left/left B-module/comodule M and for each $X \in Ob(\mathcal{C}^{\mathtt{B}})$ define a transposition $w_{M,X} : M \otimes X \to X \otimes M$ by

$$w_{M,X}(m \otimes x) := m_{(-1)}x \otimes m_{(0)}$$

for any $m \otimes x \in M \otimes X$.

Any algebra $(C, \delta_C, \varepsilon)$ in $\mathcal{C}^{\mathtt{B}}$ is a B-module coalgebra and therefore is automatically w-transpositive. We form the objects $P_\bullet(C, M)$ and $T_\bullet(C, M) := \mathrm{colim}_S P_\bullet(C, M)$ in $(\mathbf{Mod}(k)^{op}, \otimes_k^{op})$ and consider the latter as a para-cyclic module in $\mathcal{C}$. In fact $T_\bullet(C, M)$ carries a pseudo-para-cyclic $\mathtt{B}$-module structure and

$$Q_\bullet(C, M)^{op} = \mathrm{App}_\Lambda(T_\bullet(C, M)^{\mathtt{B}})$$

where $Q_\bullet(C, M)$ is the Hopf-equivariant cocyclic object defined in [10] for a B-module coalgebra C and an arbitrary B-module/comodule M. Therefore, the Hopf cyclic (co)homology of the triple (C, B, M) is defined as the cyclic (co)homology of the cocyclic k-module $C_\bullet(C, M) := k \otimes_B Q_\bullet(C, M)$.

Similarly, any coalgebra $(A, \mu_A, 1)$ in $\mathcal{C}^{\mathtt{B}}$ is a B-module algebra and therefore is automatically w-transpositive. We form the objects $P_\bullet(A, M)$ and $T_\bullet(A, M) := \mathrm{colim}_S P_\bullet(A, M)$ in $(\mathbf{Mod}(k)^{op}, \otimes_k^{op})$ and consider the latter as a para-cocyclic module in $\mathcal{C}$. In fact $T_\bullet(A, M)$ carries a pseudo-para-cocyclic $\mathtt{B}$-module structure and we see that

$$Q_\bullet(A, M)^{op} = \mathrm{App}_\Lambda(T_\bullet(A, M)^{\mathtt{B}})$$

where $Q_\bullet(A, M)$ is the Hopf-equivariant cyclic object defined in [10] for a B-module algebra A and an arbitrary B-module/comodule M. Therefore, the Hopf cyclic (co)homology of the triple (A, B, M) is defined as the cyclic (co)homology of the cyclic k-module $C_\bullet(A, M) := k \otimes_B Q_\bullet(A, M)$.

5.2. **Hopf-Hochschild homology.** Let $(\mathcal{C}, \otimes)$, M, w, B and $\mathtt{B}$ be as before. Assume A is a B-module algebra and construct $P_\bullet(A, M)$ in $\mathcal{C}$. However, since A is a B-module algebra, $P_\bullet(A, M)$ is a functor from S into $\mathcal{C}^{\mathtt{B}}$, i.e. it is a graded B-module. This time, instead of considering colimit of $P_\bullet(A, M) \to \mathcal{C}$, we consider the colimit of $P_\bullet(A, M) \colon S \to \mathcal{C}^{\mathtt{B}}$, which we denote by $T'_\bullet(A, M)$. This colimit is a little more than pseudo-para-cyclic B-module: viewed just as a simplicial object, it is actually a simplicial B-module. The Hochschild homology of $k \otimes_B T'_\bullet(A, M)$ is the Hopf–Hochschild homology of the triple (A, B, M) as constructed in [8].

5.3. **Hopf and equivariant cyclic theory of comodule (co)algebras.** Fix a commutative unital ring k and an associative/ coassociative unital/counital k-bialgebra $(B, \mu_B, \mathbb{I}, \Delta_B, \varepsilon)$. Our base category is the category of k-modules with the ordinary tensor product over k, i.e. $(\mathcal{C}, \otimes) := (\mathbf{Mod}(k), \otimes_k)$. Our base comonad in $\mathcal{C}$ is going to be $\mathtt{B} := (B \otimes \ \cdot\ )$ thus we will use the coalgebra structure on B.

The category of left B-comodules (i.e. $\mathtt{B}$-comodules in $\mathcal{C}$) is a monoidal category with respect to the ordinary tensor product of k-modules with the diagonal coaction of B on the left. Explicitly, given a pair of B-modules X and Y, the B-comodule structure on the product is given by

$$\rho(x \otimes y) := x_{(-1)}y_{(-1)} \otimes (x_{(0)} \otimes y_{(0)})$$

for any $x \otimes y \in X \otimes Y$. However, the product is not symmetric unless B is commutative but there is a braided monoidal structure if one restricts oneself to use Yetter-Drinfeld modules.

Fix a left/left B-module/comodule M and for each $X \in Ob(\mathcal{C}^{\mathtt{B}})$ define a transposition $w_{M,X} : M \otimes X \to X \otimes M$ by

$$w_{M,X}(m \otimes x) := x_{(0)} \otimes x_{(-1)}m$$

for any $m \otimes x \in M \otimes X$.

Any coalgebra $(C, \delta_C, \varepsilon)$ in $\mathcal{C}^{\mathtt{B}}$ is a B-comodule coalgebra and therefore is automatically w-transpositive. We form the objects $P_\bullet(C, M)$ and $T_\bullet(C, M) := \mathrm{colim}_S P_\bullet(C, M)$ in $(\mathbf{Mod}(k), \otimes_k)$ and consider the latter as a para-cocyclic module in $\mathcal{C}$. In fact $T_\bullet(C, M)$ carries a pseudo-para-cocyclic $\mathtt{B}$-comodule structure and

$$Q_\bullet(C, M) := \mathrm{App}_\Lambda(T_\bullet(C, M)^{\mathtt{B}})$$

is a cocyclic $\mathtt{B}$-comodule, i.e. a cocyclic B-comodule.

Similarly, any algebra $(A, \mu_A, 1)$ in the category $\mathcal{C}^{\mathtt{B}}$ is a B-comodule algebra and therefore is automatically w-transpositive. We form the objects $P_\bullet(A, M)$ and $T_\bullet(A, M) := \mathrm{colim}_S P_\bullet(A, M)$ in $(\mathbf{Mod}(k), \otimes_k)$ and consider the latter as a para-cyclic module in $\mathcal{C}$. In fact $T_\bullet(A, M)$ carries a pseudo-para-cyclic $\mathtt{B}$-comodule structure and we see that

$$Q_\bullet(A, M) := \mathrm{App}_\Lambda(T_\bullet(A, M)^{\mathtt{B}})$$

is a cyclic $\mathtt{B}$-comodule, i.e. a cyclic B-comodule. Moreover, the cyclic cohomology of the cyclic k-module $k \otimes_B Q_\bullet(A, M)$ is the bialgebra cyclic homology of a module coalgebra as defined in [9].

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